1)

$$\mathcal{F}\{af_1(t)+bf_2(t)\} = \int_{-\infty}^{\infty}(af_1(t)+bf_2(t))e^{-i\omega t}\,dt$$
$$= \int_{-\infty}^{\infty} af_1(t)e^{-i\omega t}\,dt + \int_{-\infty}^{\infty} bf_2(t)e^{-i\omega t}\,dt$$
$$= a\int_{-\infty}^{\infty} f_1(t)e^{-i\omega t}\,dt + b\int_{-\infty}^{\infty} f_2(t)e^{-i\omega t}\,dt$$
$$= a\,F_1(\omega) + b\,F_2(\omega)$$

So the Fourier transform is linear

2)

$$\mathcal{F}\{\sin(\omega_0 t)\} = \mathcal{F}\left\{\frac{1}{2i}e^{i\omega_0 t} - \frac{1}{2i}e^{-i\omega_0 t}\right\}$$
$$= \frac{1}{2i}\mathcal{F}\{e^{i\omega_0 t}\} - \frac{1}{2i}\mathcal{F}\{e^{-i\omega_0 t}\}$$
$$= \frac{1}{2i}\left(\sqrt{2\pi}\,\delta(\omega-\omega_0) - \frac{1}{2i}\sqrt{2\pi}\,\delta(\omega+\omega_0)\right)$$
$$= \frac{1}{2i}\sqrt{2\pi}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$
$$= \frac{\pi}{i}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$

3)

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } |t|<\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) = \begin{cases} \frac{\sin(t)}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{t}{T}\right)\right\} = \int_{-\infty}^{\infty}\mathrm{rect}\left(\frac{t}{T}\right)e^{-i\omega t}\,dt$$
$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\omega t}\,dt$$
$$= \left[-\frac{1}{i\omega}e^{-i\omega t}\right]_{-\frac{T}{2}}^{\frac{T}{2}}$$
$$= \frac{1}{i\omega}\left(e^{-i\frac{\omega T}{2}} - e^{i\frac{\omega T}{2}}\right)$$
$$= \frac{2}{i\omega}\left(-2i\sin\left(\frac{\omega T}{2}\right)\right)$$
$$= T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)$$

4)

$$\mathcal{F}\{u(t)e^{-at}\} = \int_{-\infty}^{\infty} u(t)e^{-at}e^{-i\omega t}\,dt$$
$$= \int_{-\infty}^{\infty} u(t)\,e^{-t(a+i\omega)}\,dt$$
$$= \int_{0}^{\infty} e^{-t(a+i\omega)}\,dt$$
$$= \frac{1}{a+i\omega}$$

5)

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)\,e^{-i\omega t}\,dt$$
$$= \int_{-\infty}^{\infty} f(u)\,e^{-i\omega(u+t_0)}\,du$$
$$= e^{-i\omega t_0}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,du$$
$$= F(\omega)\,e^{-i\omega t_0}$$

6)

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\,e^{-i\omega t}\,dt = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} f(u)e^{-i\omega\frac{u}{a}}\,du & \text{if } a>0 \\ \frac{1}{a}\int_{\infty}^{-\infty} f(u)\,e^{-i\omega\frac{u}{a}}\,du & \text{if } a<0 \end{cases}$$

$$= \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} f(u)\,e^{-i\omega\frac{u}{a}}\,du & \text{if } a>0 \\ \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)\,e^{-i\omega\frac{u}{a}}\,du & \text{if } a<0 \end{cases}$$

$$= \frac{1}{|a|}\mathcal{F}\left\{\frac{\omega}{a}\right\}$$

7)

$$\mathcal{F}\left\{e^{-(t/t_0)^2}\right\} = \sqrt{\pi}\;t_0\,e^{-(t_0\omega/2)^2}$$

?

Width of spectrum $= \omega_0 = 2/t_0$

width t_0 · width of spectrum $= t_0\omega_0 = \frac{t_0 \cdot 2}{t_0} = 2$

8) Let $f_2(t)$ be a real odd function

Then, $\mathcal{F}(f_2(t)) = F_2(-\omega)$

So we define

$$\mathcal{F}\{f_1(t)\} = \mathcal{F}\left(\tfrac{1}{2}f_1(t)\right) = \tfrac{1}{2}\mathcal{F}(f_1(t)) = \tfrac{1}{2}F_2(\omega)$$

$$F_2(-\omega) = F_2(\omega)$$

Since: $F_1(\omega) = F_1(-\omega)$ → which is the definition of a real odd function
for odd: $F_1(-\omega) = -F_1(\omega)$

9)

$$\mathcal{F}\{e^{i\omega_0 t}f(t)\} = \int_{-\infty}^{\infty} f(t)e^{i\omega_0 t}e^{-i\omega t}\,dt$$
$$= \int_{-\infty}^{\infty} f(t)\,e^{-it(\omega-\omega_0)}\,dt$$
$$= F(\omega-\omega_0)$$

10)

$$\mathcal{F}\{f(t)\} = \mathcal{F}\{u(t)\cos\omega_0 t\}$$
$$= \frac{1}{2\pi}\mathcal{F}\{u(t)\} * \mathcal{F}\{\cos\omega_0 t\}$$
$$= \frac{1}{2\pi}\left((\tfrac{1}{i\omega} + \pi\delta(\omega)) * (\pi\delta(\omega-\omega_0) + \delta(\omega+\omega_0))\right)$$

$$= \frac{1}{2}\frac{1}{i\omega} * \left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right) + \frac{\pi}{2}\delta(\omega) * (\delta(\omega-\omega_0) + \delta(\omega+\omega_0))$$

?

$$= \frac{1}{2}\left(\frac{1}{i(\omega-\omega_0)} + \frac{1}{i(\omega+\omega_0)}\right) + \frac{\pi}{2}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

!

$$= \frac{\omega}{i(\omega^2-\omega_0^2)} + \frac{\pi}{2}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

$$= \frac{i\omega}{\omega_0^2-\omega^2} + \frac{\pi}{2}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

11)

$$\mathcal{F}\{\sin\omega_0 t\} = \frac{\pi}{i}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$

$$\mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{\omega_0}\mathcal{F}\{\omega_0\cos(\omega_0 t)\} = \frac{1}{\omega_0}\mathcal{F}\left\{\frac{d\sin(\omega_0 t)}{dt}\right\}$$
$$= \frac{1}{\omega_0}i\omega\,\mathcal{F}\{\sin(\omega_0 t)\}$$
$$= \frac{1}{\omega_0}i\omega\left(\frac{\pi}{i}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)\right)$$
$$= \frac{1}{\omega_0}\pi\omega\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$
$$= \frac{1}{\omega_0}\pi\left(\omega\delta(\omega-\omega_0) - \omega\delta(\omega+\omega_0)\right)$$
$$= \frac{1}{\omega_0}\pi\left(\omega_0\delta(\omega-\omega_0) - (-\omega_0)\delta(\omega+\omega_0)\right)$$
$$= \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

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12)

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)\,d\tau\right\} = \mathcal{F}\left\{f(\tau) * u(\tau)\right\} = \mathcal{F}(f(\tau))\,\mathcal{F}(u(\tau))$$
$$= F(\omega)\left(\frac{1}{i\omega} + \pi\delta(\omega)\right)$$
$$= \frac{1}{i\omega}F(\omega) + \pi F(\omega)\delta(\omega)$$
$$= \frac{1}{i\omega}F(\omega) + \pi F(0)\delta(\omega)$$