Lecture

Friday, September 20, 2024 1:00 PM

$x(t) \to \boxed{\;T\;} \to y(t)$

$x(0)$ $\qquad y(0) = H(x(t))$

$y = T[x]$

T is the transformation that is a function

T is operator (function of functions)

$y(0)$

$f(t): t \to f(t)$ (value)

$T[x]: x \to T[x]$ (function)

two ways to add: $S_n: y(t) = x(t-n)$, $n \to S_n[x]$

multiply: $P: y(t) = x(t)\sin(t)$, $n \to P[x]$

integral: $I: \int_0^t x(t)\,dt$, $n \to I[x]$

differential: $D: y(t) = x'(t)$, $n \to D[x]$

full wave rectifier: $R: y(t) = |x(t)|$, $n \to$?

two systems in parallel: summing $\quad T = T_1 + T_2$

$T[x] = (T_1 + T_2)[x]$

$= T_1[x] + T_2[x]$

$T_1 + T_2 = T_2 + T_1$

two systems in series: $\quad T = T_1 T_2$

$x(t) \to \boxed{T_2} \to T_2[x(t)] \to \boxed{T_1} \to y(t)$

$y = T[x] = T_1 T_2 [x] = T_1[T_2[x]]$

in general $\boxed{T_1 T_2 \neq T_2 T_1}$!

two operators T_1 and T_2 commute if, for all x

$T_1 T_2 [x] = T_2 T_1 [x]$ $\quad$ common sense question : prove this

$S_1 \quad S_n[x]$

$x(t) \to x(t-(t+n))$

- time shift
- time shift and differentiation
- time shift and integration
- differentiation and integration $\quad$ only if smooth $\to$ smooth $x(t+h) = 0$

$T_1 (T_2 + T_3) \neq T_1 T_2 + T_1 T_3$

6 system properties

linearity $\to$ $T[ax(t) + \beta x(t)] = \alpha T[x(t)] + \beta T[x(t)]$

time invariance $\to$ delay is not linear to the time shift, also $\quad y(t) = T[x(t)] \Leftrightarrow y(t-\tau) = T[x(t-\tau)]$

invertibility $\to$ if there are inputs that produce same output $\quad (x_1 \neq x_2 \Rightarrow T[x_1] = T[x_2] \Leftrightarrow T[x_1] = T[x_2])$

memory $\to$ system is memoryless if $y(t)$ depends only on $x(t)$ $\quad$ otherwise

causality $\to$ system is causal if $y(t)$ depends only on $x(t)$ for $t \leq t_0$

BIBO - stability

for all t, output is bounded for bounded input

$T[x(t)] = x(t) + C$, $C \neq 0$ is not linear

memoryless system is causal

delay system is causal

advance system is non-causal $y(t) = x(t+1)$

all typical system is invertible

$y(t) = tx(t)$ is not BIBO-stable if $x(t) = 1$, the $y(t) = t$ which is unbounded

counter-example $\to$ show the claim is false for some case $\quad$ show one case you can show it yourself

integrator is not BIBO-stable $\to$ take $x(t) = 1$, the $y(t) = \int_0^t 1\,d\tau = t \to \infty$ which is unbounded

LTI $\to$ linear, time-invariant system

impulse response $h(t)$ of T

$$\boxed{h(t) = T[\delta(t)]}$$

$h(t-\tau) = T[\delta(t-\tau)]$

$y(t) = k\,x(t)$

$h(t) = k\,\delta(t)$

$y(t) = \int_0^t x(\tau)\,d\tau$

take $x(t) = \delta(t)$

$h(t) = \int_0^t \delta(\tau)\,d\tau$

$\int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,dt = x(\tau)$

$x(\tau) = \int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,dt$

$h(t) = T[\delta(t)]$

$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

If T is an LTI - system

$y(t) = T[x(t)] = T\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] = \int_{-\infty}^{\infty} T[x(\tau)\,\delta(t-\tau)]\,d\tau$

$= \int_{-\infty}^{\infty} x(\tau)\,T[\delta(t-\tau)]\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = h(t) * x(t)$

$$\boxed{y(t) = h(t) * x(t)}$$

An LTI system is completely characterized by its response to an impulse

If we know $h(t)$, we can calculate $y(t)$ for any $x(t)$

parallel system: $\quad y = T_1[x] + T_2[x] = h * x + h_2 * x = (h_1 + h_2) * x$

series system: $\quad y = T_1 T_2[x] = T_1[x * h] = h_1 * (h * x) = (h_1 * h) * x$

But $h_1 * h_2 = h * h_1 \Rightarrow T_1 T_2 = T_2 T_1 \Longrightarrow$ $\boxed{\text{LTI system commute!}}$

for a causal system: $h(t) = 0$, if $t < 0$

ideal low-pass filter

$H(\omega) = \mathrm{rect}\left(\omega / 2\omega_c\right)$

frequency domain relation

$Y(\omega) = H(\omega)\,X(\omega)$

$y(t) = h(t) * x(t)$

$H(\omega)$ is the transfer function of some final LTI system

$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$

For the ideal low-pass filter

$H(\omega) = \mathrm{rect}\left(\omega / 2\omega_c\right)$

$h(t) = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\omega_c t\right)$

$\boxed{\text{The filter is not causal, and is non-causal!}}$

$y(t) = h(t) * x(t)$

$x(t) = h_1(t) * y(t)$ $\quad$ impulse response inverse

Combination: $\quad x(t) = h_1(t) * h(t) * x(t) \Leftrightarrow h_1(t) * h(t) = \delta(t)$

Fourier transform: $\quad H_1(\omega)\,H(\omega) = 1$

$H_1(\omega)\,H(\omega) \Rightarrow H_1(\omega) = \frac{1}{H(\omega)}$

An LTI is invertible if (and only if) $H(\omega) \neq 0$

Take for ex: $h(t) = \mathcal{F}^{-1}\left(\frac{1}{\sin\omega}\right) = \mathcal{F}^{-1}\left(\frac{1}{\mathrm{rect}(\omega)}\right)$