Lecture

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$\underline{V} = V^i \underline{a}_i = \sum_i V^i \underline{a}_i$

$\underline{u} = u^j \underline{a}_j = \sum_j u^j \underline{a}_j$

$\underline{u} \cdot \underline{v} = \sum_j \sum_i v^i u^j \underline{a}_i \cdot \underline{a}_j$

$= v^i u^j a_i a_j$

$= v^i u^j g_{ij}$ metric tensor

continuity $\lim_{t \to a} |f(t) - f(a)| = 0$

$\lim_{t \to a} ||x(t) - x(a)|| = 0$

only difference: (Euclidean) length instead of absolute value

differentiation $f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$

$u'(t) = \lim_{\Delta t \to 0} \frac{u(t+\Delta t) - u(t)}{\Delta t}$

$\frac{df(t)}{dt} = \dot{f}(t) = f'(t)$

$\frac{du(t)}{dt} = \dot{u}(t) = u'(t)$

$u'(t) = 0$ if u is constant

$(\alpha u + \beta v)'(t) = \alpha u' + \beta v'$

$(f \cdot u)'(t) = f(t) \cdot u'(t) + f'(t) \cdot u(t)$

$\frac{d}{dt} u(f(t)) = \frac{du}{df} \frac{df}{dt} = u'(f) f'(t)$

prove by writing $u(t) = e_x u_x(t) + e_y u_y(t) + e_z u_z(t)$

results are 'vector expressions'; generally valid

Important for exam

$(u \cdot v)' = u' \cdot v + u \cdot v'$

$(u \times v)' = u' \times v + u \times v'$

$\underline{p} = m \underline{x}'$

$\underline{L} = \underline{x} \times m \underline{x}'$

$\underline{F} = m x''$

$\mathcal{E} = \frac{1}{2} m \underline{x}' \cdot \underline{x}'$ write the dot!

$\mathcal{E}' = \frac{1}{2} m [\underline{x}' \cdot \underline{x}']'$

$= \frac{1}{2} m [\underline{x}' \cdot \underline{x}'' + \underline{x}'' \cdot \underline{x}']$

$\frac{1}{2} m \; 2 \underline{x}' \cdot \underline{x}''$

$= \underline{x}' \cdot m \underline{x}''$

$= \underline{x}' \cdot \underline{F}$

perpendicular

$\mathcal{E}' = \underline{x}' \cdot q(\underline{x}' \times \underline{B}) = 0$

$L' = m[\underline{x}' \times \underline{x}']' = m[\underline{x}' \times \underline{x}' + \underline{x} \times \underline{x}'']$

$= m \underline{x} \times \underline{x}''$

$= \underline{x} \times m \underline{x}''$

$= \underline{x} \times \underline{F}$

$u(x^1, x^2, x^3)$

Partial derivative $\frac{\partial u}{\partial x^1}$ keep x^2, x^3 constant, take the derivative with respect to x^1

notation

$u(x^1, x^2, x^3) = u^i(x^1, x^2, x^3)\, a_i(x^1, x^2, x^3)$

$\frac{\partial u}{\partial x^1} = \frac{\partial u^i}{\partial x^1} a_i + u^i \frac{\partial a_i}{\partial x^1}$

components of derivatives $\neq$ derivatives of components

A curve is a **vector-valued** function with one parameter

parameter $t \in [t_a, t_b]$

begin point, end point

can also be open interval on one or both sides

closed curve $x(t_a) = x(t_b)$

simple curve: no crossings (except maybe in the begin/end point)

circle: $x(\phi) = \underline{e}_x (R \cos\phi) + \underline{e}_y (R \sin\phi)$ $\quad \phi \in [0, 2\pi]$

$x(t) = \underline{e}_x R \cos(2\pi t^2) + \underline{e}_y R \sin(2\pi t^2)$ $\quad t = [0, 1]$

$x(\phi) = \underline{e}_x R \cos\phi + e_y R \sin(n\phi)$ $\quad n \in \mathbb{Z}, n \geq 2 \rightarrow$ Lissajous figure

$r(\phi) = a e^{b\phi}$ Spira Mirabilis

nautilus shell, cyclone, whirlpool galaxy

$\underline{x}'(t)$ is tangential to $\underline{x}(t)$

It represents the velocity of the curve

Also, if t is not time, $\underline{x}$ is not position!

The speed is defined as $||x'(t)||$

for example given curve $v(a)$, the velocity is $\underline{v}'(a)$!

arc length of $[t_a, t]$ $\quad S(t) = \int_{t_a}^{t} ||x'(t)|| dt$

$u, \; t = t(u)$

$x_u(u) = x_t(t(u))$

$x_u'(u) = \frac{dx_t(t(u))}{dt} \frac{dt(u)}{du} = x_t'(t(u))\, t'(u)$

abs (vector) abs (number)

$||x_u'(u)|| = ||x_t'(t)|| \; |t'(u)|$

arc length s

now, consider $t(s)$

if we chose $t = t(s)$ and use the arc length s as parameter $\rightarrow$ speed is unity $\Rightarrow$ arc length parametrization

$||x_s'(s)|| = ||x_t'(t)|| \; |t'(s)| = |s'(t)| \; |t'(s)| = 1$