Lecture

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Cartesian coordinates : basis vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$

Polar coordinates : basis vectors $\vec{e}_r, \vec{e}_\varphi$

$$\underline{x}(x,y) = \underline{x}_{\varphi}(r,\varphi) = \underline{x}(x(r,\varphi), y(r,\varphi))$$

In physics, 'assume': $\underline{x}(x,y) = \underline{x}(r,\varphi)$

$$r_p(r,\phi) = r(x(r,\phi), y(r,\phi)) = \vec{e}_x r\cos\phi + \vec{e}_y r\sin\phi$$

$$\vec{e}_r = C_r \left.\frac{\partial\vec{r}}{\partial r}\right|_\phi = C_r(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y)$$

$$= \frac{1}{\sqrt{\cos^2\phi + \sin^2\phi}}(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y)$$

$$= \cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y$$

$$\vec{e}_\phi = C_\phi \left.\frac{\partial\vec{r}}{\partial\phi}\right|_r = C_\phi(-r\sin\phi\,\vec{e}_x + r\cos\phi\,\vec{e}_y) = -\sin\phi\,\vec{e}_x + \cos\phi\,\vec{e}_y$$

$$C_\phi = \frac{1}{\sqrt{(r\sin\phi)^2 + (r\cos\phi)^2}} = \frac{1}{\sqrt{r^2}} = \frac{1}{r}$$

$$r(r,\phi) = r\,\vec{e}_r(\varphi)$$

a vector is a collection of components and basis vectors!

define $\vec{a}_i = \frac{\partial r}{\partial x^i}$

$$\frac{d\vec{r}}{dt} = \frac{dx^i}{dt}\vec{a}_i = \sum_{i=1}^{3}\frac{\partial\vec{r}}{\partial x^i}\frac{dx^i}{dt}$$

$(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ forms the covariant basis for (x_1, x_2, x_3)

$$\vec{a}_i = \frac{\partial\vec{x}}{\partial x^i}$$

covariant basis : tangential to the coordinate lines
- not necessarily orthogonal
- not necessarily normalized

$$\text{speed} = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = \sqrt{\frac{dx^i}{dt}\vec{a}_i\cdot\frac{dx^j}{dt}\vec{a}_j} = \sqrt{\frac{dx^i}{dt}\frac{dx^j}{dt}a_i\cdot a_j} = \sqrt{\frac{dx^i}{dt}\frac{dx^j}{dt}g_{ij}}$$

metric tensor $g_{ij} = \vec{a}_i\cdot\vec{a}_j$

$$\underline{\underline{g}} = \begin{bmatrix} \vec{a}_1\cdot\vec{a}_1 & \vec{a}_1\cdot\vec{a}_2 & \vec{a}_1\cdot\vec{a}_3 \\ \vec{a}_2\cdot\vec{a}_1 & \vec{a}_2\cdot\vec{a}_2 & \vec{a}_2\cdot\vec{a}_3 \\ \vec{a}_3\cdot\vec{a}_1 & \vec{a}_3\cdot\vec{a}_2 & \vec{a}_3\cdot\vec{a}_3 \end{bmatrix}$$ symmetric

ortho - curvilinear systems (OCL): $a_i\cdot a_j = 0$ if $i \neq j$

$h_i = \|a_i\|$: lengths of covariant basis vectors (scale factors/Lamé coefficients)

$$g_{ii} = h_i^2$$

$$\left\|\frac{dr}{dt}\right\| = \sqrt{\left(h_i\frac{dx_i}{dt}\right)^2}$$

$$d\vec{r} = \frac{\partial\vec{r}}{\partial x^i}dx^i = \vec{a}_i\,dx^i$$

$$ds = \sqrt{\vec{a}_i\,dx^i\cdot\vec{a}_j\,dx_j} = \sqrt{g_{ij}\,dx^i dx^j}$$

in OCL coordinates:

$$ds = \sqrt{\sum_i (h_i\,dx^i)^2}$$