Lecture

Tuesday, 7 January 2020 13:33

∇f	gradient	number (scalar) → vector
$\nabla \cdot \underline{v}$	divergence	vector → number (scalar)
$\nabla \times \underline{v}$	curl	vector → vector

$$\frac{d fg}{dt} = f\frac{dg}{dt} + \frac{df}{dt}g$$

$$\nabla fg = \frac{\partial fg}{\partial x^i}\underline{a}^i = \left[f\frac{\partial g}{\partial x^i} + g\frac{\partial f}{\partial x^i}\right]\underline{a}^i = f\nabla g + (\nabla f)g$$

$$\nabla \cdot (\nabla \times \underline{a}) = 0$$

$$\nabla \times \nabla f = \underline{0}$$

Write the line below or arrow above!

$$\nabla \cdot \left(f \; \nabla \times \underline{v}\right) = f\big(\underbrace{\nabla \cdot (\nabla \times \underline{v})}_{=0}\big) + \nabla f \cdot (\nabla \times \underline{v}) = \nabla f \cdot (\nabla \times \underline{v})$$

$$\nabla \cdot (\underbrace{\nabla f}_{\underline{a}} \times \underbrace{\nabla g}_{\underline{b}}) = \underbrace{(\nabla \times \nabla f)}_{=0} \cdot \nabla g - \nabla f \cdot \underbrace{(\nabla \times \nabla g)}_{=0} = \underline{0}$$

$$\nabla^2 f = \Delta f = \nabla \cdot \nabla f$$

$$\nabla f = \frac{\partial f}{\partial x^j}\underline{a}^j$$

$$\nabla \cdot \underline{v} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, v^i\right)$$

$$v^i = g^{ij} v_j = v_j g^{ij}$$

$$g^{ij}\frac{\partial f}{\partial x^j}$$

$$\nabla^2 f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\frac{\partial f}{\partial x^j}\right)$$

$$\nabla^2 f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\frac{\partial f}{\partial x^j}\right)$$

$$\text{OCL}: g_{ij} = \begin{pmatrix} h_1^2 & & \\ & h_2^2 & \\ & & h_3^2 \end{pmatrix} = \left[\underline{a}_i \cdot \underline{a}_j\right]$$

$$g^{ij} = \begin{pmatrix} h_1^{-2} & & \\ & h_2^{-2} & \\ & & h_3^{-2} \end{pmatrix}$$

$$\sqrt{g} = \sqrt{\det g_{ij}} = h_1 h_2 h_3$$

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\sum_{ij}\frac{\partial}{\partial x^i}\left(h_1h_2h_3\begin{pmatrix} h_1^{-2} & 0 & 0 \\ 0 & h_2^{-2} & 0 \\ 0 & 0 & h_3^{-2}\end{pmatrix}\frac{\partial f}{\partial x^j}\right)$$

$$= \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x^1}\left(h_1h_2h_3\, h_1^{-2}\frac{\partial f}{\partial x^1}\right)\right)$$

$$+ \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x^2}\left(h_1h_2h_3 \cdot h_2^{-2}\frac{\partial f}{\partial x^2}\right)\right)$$

$$+ \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x^3}\left(h_1h_2h_3 \cdot h_3^{-2}\frac{\partial f}{\partial x^3}\right)\right)$$

because $g^{ij} = 0$ when $i \neq j$, and we are summing over all i and j

$$= \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x^1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial x^3}\right)\right)$$

$$= \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ii}\frac{\partial f}{\partial x^i}\right)$$

$$= \sum_i \frac{1}{h_1h_2h_3}\frac{\partial}{\partial x^i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial f}{\partial x^i}\right)$$

~~$\nabla^2(f(\underline{x})g(\underline{x})) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\frac{\partial fg}{\partial x^j}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\left(\frac{\partial f}{\partial x^j}g + \frac{\partial g}{\partial x^j}f\right)\right)$~~

$$\nabla^2(f(\underline{x})g(\underline{x})) = \nabla \cdot (f\nabla g + g\nabla f)$$

$$= f\nabla^2 g + \nabla f \cdot \nabla g + g\nabla^2 f + \nabla g \cdot \nabla f$$

$$= f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$$

↳ = 0 if and only if gradients orthogonal?!

$$\nabla^2 f(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \;\ldots\ldots$$

$$= \frac{-\alpha}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0 \text{ for } r \neq 0$$