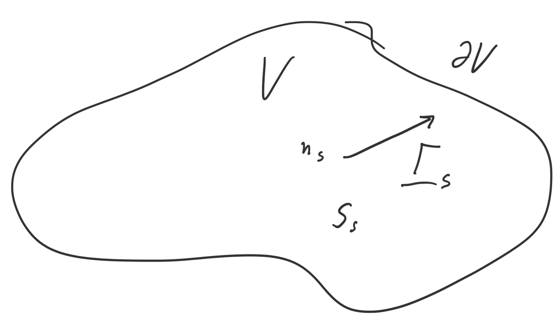
Lecture

Thursday, 9 January 2020 08:49

Charge continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{j} = 0$$



$$\frac{\partial n_s}{\partial t} + \nabla \cdot \underline{\Gamma}_s = S_s$$ ← must be able to derive

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{j} = \Sigma\, q_s S_s = 0$$ ← total charge does not change

$$\delta(\underline{x} - \underline{x}_0) = \underline{0} \quad \text{if } \underline{x} \neq \underline{x}_0$$

$$\int f(\underline{x})\, \delta(\underline{x} - \underline{x}_0)\, dV = f(\underline{x}_0)$$

point charge at $x = x_0$

$$\rho(\underline{x}) = q\, \delta(\underline{x} - \underline{x}_0)$$

$$\int \rho(\underline{x})\, dV = q \int \delta(\underline{x} - \underline{x}_0)\, dV = q$$

$$\underline{E}(\underline{x}(r, \theta, \phi)) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \underline{e}_r = \frac{1}{4\pi\varepsilon_0} q \frac{\underline{x}}{\|\underline{x}\|^3}$$

$$\phi(\underline{x}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\|\underline{x}\|}$$

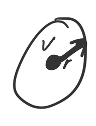
$$\underline{E} = -\nabla \phi$$

$$\nabla \cdot \underline{E} = -\nabla^2 \phi = 0 \quad \text{for } r \neq 0$$

proposition:

$$\nabla \cdot \underline{E} = \alpha\, \delta(\underline{x})$$

What is α? Integrate E over a sphere.



$$\int_V \nabla \cdot \underline{E}\, dV = \alpha \int_V \delta(\underline{x})\, dV = \alpha$$

$$= \int_{\partial V} \underline{E} \cdot \underline{n}\, dA = E_r(r) \cdot 4\pi r^2 = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0} q\, \delta(\underline{x}) = \frac{1}{\varepsilon_0} \rho(\underline{x}), \quad \rho(x) = q\, \delta(x)$$

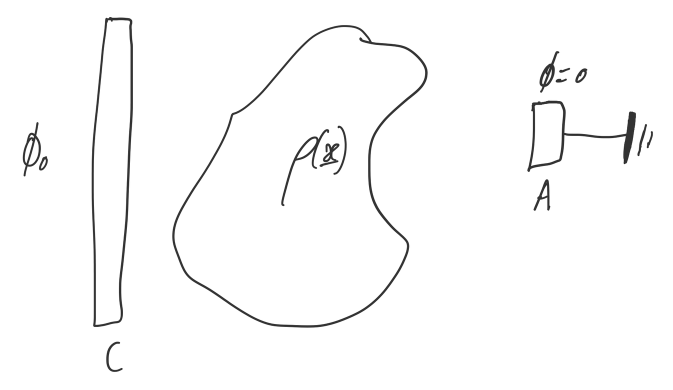
$$\boxed{\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}}$$

$$\underline{E} = -\nabla \phi$$

$$\rightarrow \nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}}$$

$$d\phi = \frac{1}{4\pi\varepsilon_0} \frac{\rho(\underline{x}')}{\|\underline{x} - \underline{x}'\|} dV$$

$$\phi(\underline{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\underline{x}')}{\|\underline{x} - \underline{x}'\|} dV$$



$$\nabla \cdot \underline{B} = 0$$

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t}$$

$$\boxed{\nabla \times \underline{B} = \mu_0 \underline{j}}$$ → derived for constant current

$$\nabla \cdot \nabla \times \underline{B} = 0 = \mu_0 \nabla \cdot \underline{j}$$

$$= -\mu_0 \frac{\partial \rho}{\partial t}$$

$$\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \nabla \times \underline{B} = 0 = \mu_0 \nabla \cdot \underline{j} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \underline{E}$$

$$= \mu_0 \nabla \cdot \underline{j} + \mu_0 \frac{\partial \rho}{\partial t}$$

$$= -\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \frac{\partial \rho}{\partial t}$$

$$= 0$$

$$\underline{j} + \varepsilon_0 \frac{\partial E}{\partial t}$$

$$0 = \nabla \cdot \nabla \times \underline{E} = -\nabla \cdot \frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla \cdot \underline{B})$$

if $(\nabla \cdot B)_{t_0} = \underline{0}$ then $\nabla \cdot B = 0$ forever

potential

$$\underline{V} = -\nabla \phi$$

$$\int_A^B \underline{V} \cdot d\underline{x} = \phi(\underline{x}_B) - \phi(\underline{x}_A)$$

Condition for potential existence: $\nabla \times \underline{V} = \underline{0}$

$$\nabla \times \nabla \phi = 0$$

$$\nabla \cdot \nabla \times \underline{V} = 0$$

if $\nabla \cdot \underline{V} = 0 \longrightarrow \underline{V} = \nabla \times \underline{A}$

$\nabla \times \underline{V} = 0 \rightarrow$ curl-free / irrotational

$$\nabla \times (\nabla \phi) = \underline{0}$$

$$\underline{V} = -\nabla \phi$$

$\nabla \cdot \underline{V} = 0 \rightarrow$ divergence-free / source-free / solenoidal

$$\nabla \cdot (\nabla \times \underline{A}) = 0$$

$$\underline{V} = \nabla \times \underline{A}$$

Helmholtz Decomposition Theory

Every vector field can be written as

$$\underline{V} = -\nabla \phi + \nabla \times A$$

$$\phi(\underline{x}) = \frac{1}{4\pi} \int_V \frac{\nabla' \cdot \underline{V}(\underline{x}')}{\|\underline{x} - \underline{x}'\|} dV' - \frac{1}{4\pi} \oint_{\partial V} \frac{\underline{n}' \cdot \underline{V}(\underline{x}')}{\|\underline{x} - \underline{x}'\|} dS'$$

$$\underline{A}(\underline{x}) = \frac{1}{4\pi} \int_V \frac{\nabla' \times \underline{V}(\underline{x}')}{\|\underline{x} - \underline{x}'\|} dV' - \frac{1}{4\pi} \oint_{\partial V} \frac{\underline{n}' \cdot \underline{V}(\underline{x}')}{\|\underline{x} - \underline{x}'\|} dS'$$